On the Stability of Equilibrium Sets of Discrete Conservative Mechanical Systems

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A generalized version of Lagrange-Dirichlet's stability theorem is presented for systems possessing a continuous set of equilibrium positions. The notion of "stability of a set" is used and the theorem is given in a form analogous to Andronov and Witt's theorem on the orbital stability of a periodic solution of an autonomous system. The theorem also holds true for steady motions in systems with cyclic coordinates. As an application, the steady motions of a dual spin satellite are discussed.

Introduction

MOST theorems on the stability of the equilibrium of a mechanical system apply to isolated equilibria only. However, in a number of problems there are continuous sets of equilibrium positions rather than isolated points. For the study of their stability properties it is then convenient to use the notion of "stability of a set" (see e.g., Zubov,[1] Yoshizawa[2]). A generalized version of Lagrange-Dirichlet's stability theorem can be applied to systems which possess m-parametric families of equilibrium positions. Moreover, it can be put in a form which is analogous to Andronov and Witt's theorem on the orbital stability of periodic solutions of autonomous systems (see Malkin,[3] p. 259, for the more general version of this theorem). The theorem on the stability of the equilibrium set then asserts that a sufficient condition for the stability of an m-parametric family of equilibria is that the matrix $(U_{q_iq_j})$ has n-m positive eigenvalues at each point of the equilibrium set, n being the number of degrees of freedom, $U(\mathbf{q})$ the potential energy and q_i, $i = 1,2\ldots n$ the generalized coordinates. In the paper, the stability theorem is formulated and proved. Some other results are then discussed and two applications are given, one of these being of technical interest.

Stability Theorem

Consider a discrete autonomous and holonomic conservative mechanical system with n degrees of freedom. Its generalized coordinates shall be $q_1, q_2, \ldots, q_n$, in matrix representation

$$\mathbf{q} = (q_1, \ldots, q_n)^T$$

The system is characterized by the kinetic energy function

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \tfrac{1}{2}\dot{\mathbf{q}}^T \mathbf{A}(\mathbf{q})\dot{\mathbf{q}}$$

where $\mathbf{A}(\mathbf{q}) \in C_1$ is positive definite for all values of $\mathbf{q} \in \Omega = \{\mathbf{q} | a_i \leqq q_i \leqq b_i, i = 1,2,\ldots,n\}$, and by the potential energy function $U(\mathbf{q}) \in C_3, \forall \mathbf{q} \in \Omega$. The equations of motion are given by

$$(d/dt)(\partial L/\partial \dot{q}_i) - \partial L/\partial q_i = 0, \qquad i = 1,2,\ldots,n$$

with $L = T - U$, and can always be put into the form

$$\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) \tag{1}$$

We assume that the system possesses an m-parametric family of equilibrium positions, given by $M' = \{\mathbf{q} | \mathbf{q}(\alpha_1, \alpha_2, \ldots, \alpha_m),\ c_i \leqq \alpha_i \leqq d_i, i = 1,2,\ldots,m\}$ with $M' \subset \Omega$, where $\mathbf{q}(\alpha_1, \alpha_2, \ldots, \alpha_m) \in C_1$. At each point $\bar{\mathbf{q}} \in M'$ the set M' is presumed to be continuously differentiable and to have an uniquely determined m-dimensional tangent linear manifold $T(\bar{\mathbf{q}})$ in the configuration space. This implies that

$$\operatorname{rank}(\partial q_i/\partial \alpha_j) = m \forall\ \bar{\mathbf{q}} \in M'$$

The tangent manifold M' is spanned by the vectors $\mathbf{g}_i$ defined as

$$\mathbf{g}_i = \frac{\partial \mathbf{q}}{\partial \alpha_i} \Big/ \left|\frac{\partial \mathbf{q}}{\partial \alpha_i}\right|, \quad i = 1,2,\ldots,m$$

The set $M = M' \times \mathbf{O}_n$ is then clearly an invariant set of the differential Eq. (1) (see e.g., Zubov,[1] p. 21, Yoshizawa,[2] p. 71, etc.). In what follows we will discuss the stability of M.

Theorem

If the matrix $[U_{q_iq_j}(\bar{\mathbf{q}})]$ has n-m strictly positive eigenvalues at each point $\bar{\mathbf{q}} \in M'$, then the set M is stable.

Proof

We know that $\mathbf{grad}\, U = \mathbf{O}, \forall \bar{\mathbf{q}} \in M'$. Therefore, U is constant on M'. Let us assume, without loss of generality, that $U = 0$ on M'. We then have

$$U(\mathbf{q}) = \tfrac{1}{2}(\mathbf{q} - \bar{\mathbf{q}})^T [U_{q_iq_j}(\bar{\mathbf{q}})] (\mathbf{q} - \bar{\mathbf{q}}) + o(|\mathbf{q} - \bar{\mathbf{q}}|^2)$$

$U_{q_iq_j}$ being symmetric, the eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ can be chosen orthonormal. Consider the manifold $N(\bar{\mathbf{q}})$ spanned by the eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-m}$ of $[U_{q_iq_j}(\bar{\mathbf{q}})]$ corresponding to the n-m positive eigenvalues. Let us show that $N(\bar{\mathbf{q}}) = T^{\perp}(\bar{\mathbf{q}})$.

Suppose $N(\bar{\mathbf{q}}) \neq T^{\perp}(\bar{\mathbf{q}})$ and consider an orthonormal basis $\{\mathbf{t}_1, \mathbf{t}_2 \ldots \mathbf{t}_m\}$ of $T(\bar{\mathbf{q}})$. Then there must exist some $\mathbf{t}_j, j = 1,2,\ldots,m$ and some $\mathbf{e}_i$, $i = 1,2,\ldots,n$-m such that $\langle \mathbf{e}_i, \mathbf{t}_j \rangle = a_{ij} \neq 0$. Let $\bar{\bar{\mathbf{q}}} \neq \bar{\mathbf{q}}$ be some point on M' such that

$$\bar{\bar{\mathbf{q}}} - \bar{\mathbf{q}} = |\bar{\bar{\mathbf{q}}} - \bar{\mathbf{q}}|\mathbf{t}_j + 0(\bar{\bar{\mathbf{q}}} - \bar{\mathbf{q}}|)$$

then

$$U(\bar{\bar{\mathbf{q}}}) = 0 = \tfrac{1}{2}(\bar{\bar{\mathbf{q}}} - \bar{\mathbf{q}})^T [U_{q_iq_j}(\bar{\mathbf{q}})] (\bar{\bar{\mathbf{q}}} - \bar{\mathbf{q}}) + o(|\bar{\bar{\mathbf{q}}} - \bar{\mathbf{q}}|^2)$$

and

$$(\bar{\bar{\mathbf{q}}} - \bar{\mathbf{q}})^T [U_{q_iq_j}(\bar{\mathbf{q}})] (\bar{\bar{\mathbf{q}}} - \bar{\mathbf{q}}) = o(|\bar{\bar{\mathbf{q}}} - \bar{\mathbf{q}}|^2)$$

With $|\bar{\bar{\mathbf{q}}} - \bar{\mathbf{q}}| = s$, we have

$$\bar{\bar{\mathbf{q}}} - \bar{\mathbf{q}} = s\mathbf{t}_j + o(s)$$

and, writing the matrices corresponding to $(U_{q_iq_j})$ and $\mathbf{t}_j$ in the same basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, we easily obtain

$$o(s^2) = s\mathbf{t}_j^T [U_{q_iq_j}(\bar{\mathbf{q}})]\, s\mathbf{t}_j \geqq \lambda_i s^2 a_{ij}^2$$

where $\lambda_i > 0$ is the eigenvalue associated with $\mathbf{e}_i$. From this contradiction we conclude that there is no $\mathbf{e}_i, i = 1,2,\ldots,n$-$m$, such that $\langle \mathbf{e}_i, \mathbf{t}_j \rangle = a_{ij} \neq 0$ and therefore $N(\bar{\mathbf{q}}) = T^{\perp}(\bar{\mathbf{q}})$.

We next want to prove that there is a certain δ-neighborhood $S(M', \delta)$ of M' in which $U(\mathbf{q})$ is strictly positive. Consider some

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$\mathbf{q} \in T^{\perp}(\bar{\mathbf{q}})$. We know that the restriction of $U(\mathbf{q})$ to $T^{\perp}(\bar{\mathbf{q}})$ is positive definite, i.e., there exists a certain $\rho(\bar{\mathbf{q}}) > 0$ such that

$$U(\mathbf{q}) > 0, \quad \mathbf{q} \in T^{\perp}(\bar{\mathbf{q}}), \quad 0 < |\mathbf{q}-\bar{\mathbf{q}}| \leqq \rho(\bar{\mathbf{q}})$$

(see, e.g., Malkin,[3] p. 17). If we knew that $\rho(\bar{\mathbf{q}})$ were continuous we could simply take $\delta = \min_{\bar{\mathbf{q}} \in M'} \rho(\bar{\mathbf{q}})$. However, the continuity of $\rho(\bar{\mathbf{q}})$ is not guaranteed. Therefore, we shall prove the existence of the neighborhood $S(M', \delta)$ in a somewhat different way. Here we will use the property $U(\mathbf{q}) \in C_3$. For $\mathbf{q}-\bar{\mathbf{q}} \in N(\bar{\mathbf{q}})$, we write

$$U(\mathbf{q}) = U_2 + \mathscr{R}$$

with

$$U_2 = \frac{1}{2} \sum_{i,j=1,2\ldots n} \frac{\partial^2 U(\bar{\mathbf{q}})}{\partial q_i \partial q_j} (q_i - \bar{q}_i)(q_j - \bar{q}_j)$$

and

$$\mathscr{R} = \frac{1}{6} \sum_{i,j,k=1,2\ldots n} \frac{\partial^3 U[\bar{\mathbf{q}} + \theta(\mathbf{q}-\bar{\mathbf{q}})]}{\partial q_i \partial q_j \partial q_k} (q_i - \bar{q}_i)(q_j - \bar{q}_j)(q_k - \bar{q}_k)$$

$$0 \leqq \theta \leqq 1$$

With $s = |\mathbf{q}-\bar{\mathbf{q}}|$ we have $U_2 \geqq \frac{1}{2}\lambda_{\min}(\bar{\mathbf{q}})s^2$, where $\lambda_{\min}(\bar{\mathbf{q}})$ is the smallest positive eigenvalue of $(U_{q_i q_j})$ and

$$\mathscr{R} \geqq -\frac{s^3}{6} \max_{i,j,k} \left| \frac{\partial^3 U[\bar{\mathbf{q}} + \theta(\mathbf{q}-\bar{\mathbf{q}})]}{\partial q_i \partial q_j \partial q_k} \right|.$$

The eigenvalues are continuous functions of $\bar{q}$ (see, e.g., Ref. 4), so that

$$\lambda = \min_{\bar{\mathbf{q}} \in M'} \lambda_{\min}(\bar{\mathbf{q}}) > 0$$

With

$$\alpha(\bar{\mathbf{q}}, s) = \max_{\substack{(\mathbf{q}-\bar{\mathbf{q}}) \in N(\bar{\mathbf{q}}) \\ d(\mathbf{q}, M') \leqq s}} \left| \frac{\partial^3 U(\mathbf{q})}{\partial q_i \partial q_j \partial q_k} \right|$$

and

$$\alpha(s) = \max_{\mathbf{q} \in M'} \alpha(\bar{\mathbf{q}}, s)$$

we finally obtain

$$U_2 \geqq \tfrac{1}{2}\lambda s^2 \quad \text{and} \quad \mathscr{R} \geqq -(s^3/6)\alpha(s)$$

Therefore

$$U(\mathbf{q}) \geqq \tfrac{1}{2}s^2[\lambda - \tfrac{1}{3}s\alpha(s)], \quad (\mathbf{q}-\bar{\mathbf{q}}) \in N(\bar{\mathbf{q}}), \quad s = |\mathbf{q}-\bar{\mathbf{q}}|$$

The function $\alpha(s)$ is positive nondecreasing so that $s\alpha(s)$ is strictly monotonically increasing. We take $\delta > 0$ and smaller than the smallest root of $\lambda - \frac{1}{3}s\alpha(s) = 0$. It is then clear that $U(\mathbf{q})$ is not only strictly positive in the open δ-neighborhood $S(M', \delta)$ of M' but also that it has a "uniform lower bound."

The stability theorem is now easily proved by applying Zubov's theorem 12 (Ref. 1, p. 41) with $H = T + U$ as a Liapunov function. This function satisfies the conditions of the theorem in $S(M', \delta) \times R^n$ with $M = M' \times \mathbf{O}_n$. The existence of a uniform upper bound on $U(\mathbf{q})$, required by Zubov's theorem, can be proved exactly in the same way as the lower bound. The proof of our theorem is therefore complete.

It is worthwhile to observe that from Zubov's theorem nothing can be said in general about the stability of the individual equilibria belonging to the continuum, but only about the stability of the whole set. Mechanical systems, however, are of a very particular type, because they are derived from a Lagrangian function. In these systems everything seems to indicate that the individual equilibria of the sets M under consideration are *always* unstable.

No proof of this statement is known to the author. The statement can however be made plausible. Consider the class $\mathscr{K}$ of functions $U(\mathbf{q})$ so that the following lemma holds true: If the system (1) with $U \in \mathscr{K}$ has an equilibrium point at $\mathbf{q} = \mathbf{q}_e$ (isolated or not) and if U does not assume a strict minimum at $\mathbf{q} = \mathbf{q}_e$, then the equilibrium is unstable. It is not known for which class $\mathscr{K}$ this lemma is true; counterexamples even of class C_∞ are known (for a discussion see Ref. 5). The lemma is, however, verified by all functions U sufficiently well behaved to be of "technical interest." Then, of course, for all sufficiently well behaved functions $U(\mathbf{q})$ the individual points of the set M are unstable, because U does not have a strict minimum at any $\mathbf{q} \in M'$.

Additional Results

In the same way as Lagrange-Dirichlet's theorem has just been generalized to the stability of a set of equilibria, Routh's theorem can be generalized to the case of a continuous set of steady motions. After the elimination of cyclic coordinates, the system is characterized by Routh's function

$$R(\mathbf{q}, \dot{\mathbf{q}}, \beta) = T_2(\mathbf{q}, \dot{\mathbf{q}}) + T_1(\mathbf{q}, \dot{\mathbf{q}}, \beta) + T_0(\mathbf{q}, \beta) - U(\mathbf{q})$$

where $\beta = (\beta_i, \ldots, \beta_l)^T$ is the matrix of the momenta corresponding to the cyclic impulses (see, for example, Ref. 5, p. 306 for the explicit determination of T_2, T_1, T_0) and $\mathbf{q}$ is now the matrix of the noncyclic coordinates. Steady motions exist if the equation

$$\left. \frac{\partial W(\mathbf{q}, \beta)}{\partial q_i} \right|_{\mathbf{q}=\mathbf{q}_e} = 0, \quad \cdot = 1, 3, \ldots, n$$

has some solution $\mathbf{q}_e$, where

$$W(\mathbf{q}, \beta) = U(\mathbf{q}) - T_0(\mathbf{q}, \beta)$$

is the "dynamic potential." If $\mathbf{q}_e$ is isolated, Routh's theorem says that the corresponding steady motion is stable for perturbations in the initial conditions that leave the impulses $\beta_1, \beta_2 \ldots \beta_l$ unaltered provided the function $W(\mathbf{q}, \beta)$ assumes a minimum at $\mathbf{q} = \mathbf{q}_e$ ("reduced stability"). In 1953, Salvadori[6] showed that the steady motion is then also stable with respect to perturbations which may change β, and the same result was obtained independently by Pozharitskii[7] in 1958. This generalization of Routh's theorem is by no means trivial (see discussion in Ref. 5, p. 308).

The main difference between the cases treated by the theorems of Lagrange-Dirichlet and Routh is that in the first case a maximum of $U(\mathbf{q})$ implies instability, whereas in the second case one may very well have stable steady motions with the absence of a minimum and even with a maximum of $W(\mathbf{q}, \beta)$, due to the action of the gyroscopic terms.

If we have a continuum of solutions $\mathbf{q}_e$ (with the same β) in

$$\left. \frac{\partial W(\mathbf{q}, \beta)}{\partial q_i} \right|_{\mathbf{q}=\mathbf{q}_e} = 0, \quad i = 1, 2, \ldots, n$$

rather than isolated solutions, then again the concept of "stability of a set" is useful. If M' is the m-dimensional set of $\mathbf{q}_e$, then the set $M' \times \mathbf{O}_n$ is stable if the matrix $[W_{q_i q_j}(\mathbf{q}, \beta)]$ has n-m positive eigenvalues. The stability here is in the sense defined in Ref. 5 and the proof of this theorem is analogous to that given in the previous section, where W is now used in place of U and the Liapunov function corresponds to Jacobi's integral $T_2 + U - T_0 = h$. Moreover, the sets of steady motions are then stable even for perturbations in the initial conditions which change β, because Salvadori's argument still holds true. On the other hand, the individual steady motions of a continuous set may now very well be stable, in opposition to the case of the stability of equilibria.

Besides this generalization to steady motions, several other theorems can be proved with regard to stability of equilibrium. For instance if the matrix $[U_{q_i q_j}(\bar{\mathbf{q}})]$ has at least one negative eigenvalue for some $\bar{\mathbf{q}} \in M'$, then the set M is unstable. This is a generalization of a theorem due to Liapunov, which is given e.g., in Malkin,[3] p. 57. Obviously, nothing can be said about the stability of M if the matrix $[U_{q_i q_j}(\bar{\mathbf{q}})]$ has no negative eigenvalue for any $\bar{\mathbf{q}} \in M'$ and if simultaneously there is some point $\bar{\mathbf{q}} \in M'$ at which the matrix has more than m zero eigenvalues. Stability then depends on terms of higher order.

Similarly, if the system is not conservative, but if there are dissipative forces $Q_i(\mathbf{q}, \dot{\mathbf{q}})$ with $Q_i(\mathbf{q}, \mathbf{O}) = 0$ and

$$\sum_{i=1}^{n} \dot{q}_i Q_i(\mathbf{q}, \dot{\mathbf{q}}) \leqq 0$$

so that the equations of motion now are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i, \quad i = 1, 2, \ldots, n$$

then the stability theorem (and also the instability theorem) remains valid. If the dissipation function

$$\sum_{i=1}^{n} \dot{q}_i Q_i(\mathbf{q}, \dot{\mathbf{q}})$$

is not only semidefinite but definite in $\dot{\mathbf{q}}$, then the set M is not only stable but even asymptotically stable, provided the conditions of the first theorem are fulfilled. This follows from a generalization of La Salle's theorem (Ref. 8, p. 58). If La Salle's theorem is not used, then the proof becomes more involved, as can be seen by the analogous theorem given by Gantmacher (Ref. 9, p. 178).

Application

A dual-spin or gyrostat satellite is formed by a rigid body carrying a constant speed symmetric rotor mounted on it, the axis of the rotor being aligned either with the principal axis with least or greatest moment of inertia I_2, with $I_2 > I_3 > I_1$ or $I_1 > I_3 > I_2$ (the rotor masses are included in I_1, I_2, I_3). We assume that the satellite is in a circular orbit in a perfect inverse square law gravitational force field and that its linear dimensions are much smaller than its mean distance to the Earth. Then the equations which give the satellites orbit around the Earth are assumed to be independent from the equations which describe its orientation in space. This system has three noncyclic coordinates which describe the orientation of the satellite in its orbit. In general, the satellite is capable of a discrete set of steady motions for which one side of the body faces toward the Earth at all times. However, Longman[10] has shown that if the relation

$$H = 2\omega_0[(I_3 - I_2)(I_1 - I_2)]^{1/2}$$

is satisfied, H being the angular momentum of the rotor relative to the main body and ω_0 the orbital angular velocity, then the steady motions or "relative equilibria" are no longer isolated but form two closed continuous curves in the space of configuration.[10] The set M' is now no longer simply connected but formed by two disconnected closed curves. We could study the stability of one of these curves separately; however, as they are symmetric, the result will be the same as if we consider M' as formed by the sum of these two disjoint sets. The parametric equations of M' will not be reproduced here. Longman[10] has examined more closely the continuum of steady motions and it can be seen that it really is formed by two closed lines satisfying all the regularity conditions required for the application of the stability theorem. In Ref. 10 the dynamic potential $W(\omega_0, \bar{\mathbf{q}})$ is written as a function of the Euler angles $\theta_1, \theta_2, \theta_3$ and the following characteristic equation is obtained for the matrix $(W_{\theta_i \theta_j})$ on M':

$$\lambda\{\lambda^2 - [5(I_3 - I_2)/I_2 - (4x-5)(I_1 - I_2)/I_2]\lambda + [4x^2 - 10(K+1)x + (4K^2 + 17K + 4)](I_1 - I_2)^2/I_2^2\} = 0$$

Here K is given by $K = (I_3 - I_2)/(I_1 - I_2)$ with $0 < K < 1$ and the parameter x is the same parameter which appears in the parametric equations of the closed lines forming M', its range being $K \leq x \leq \min(4K, 1)$. The characteristic equation has always one zero root $\lambda_1 = 0$. The signs of the other roots λ_2, λ_3 are at all points of the continuum M' given by Table 1. But $n - r = 3$ and $m = 1$ and therefore the set of steady motions is stable if the rotor axis is aligned with the axis of minimum moment of inertia of the body and if simultaneously $K \neq \frac{1}{4}$. A family of tumbling motions around the set of steady motions or around one of the steady motions then exists. In all the other cases we do not know whether or not M is stable.

These results were obtained by Longman by means of a general discussion. An interpretation in terms of Liapunov's method was also given. Here, a formal proof is given by the application of our stability theorem and using the notion of "stability of a set."

Table 1 Roots of the characteristic equation

Value of K	Relation between I_1, I_2, I_3	λ_1	λ_2
$\neq \frac{1}{4}$	$I_1, I_3 > I_2$	> 0	> 0
$\neq \frac{1}{4}$	$I_1, I_3 < I_2$	< 0	< 0
$= \frac{1}{4}$	$I_1, I_3 > I_2$	$= 0$	> 0
$= \frac{1}{4}$	$I_1, I_3 < I_2$	$= 0$	< 0

We obtain a more general result than Longman if we keep in mind that the theorem on the stability of steady motions is valid in its most general form, analogous to Salvadori's theorem. Then we see immediately that the relative equilibrium continuum is stable even with respect to perturbations in the initial conditions which change the momentum corresponding to the cyclic coordinates giving rise to a new circular orbit. This means that the angular velocity ω_0 is changed so that the relation

$$H = 2\omega_0[(I_3 - I_2)(I_1 - I_2)]^{1/2}$$

is no longer fulfilled. Although there are then only 24 isolated relative equilibria and there is no longer a continuum, we know that the motions remain close to the original continuum for all times, provided only that the initial conditions are sufficiently close to it; it is not necessary that the initial conditions be close to one of the 24 isolated equilibria associated with the new ω_0.

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